hep-th/0405045

The Final State of Black Strings and p-Branes, and the Gregory-Laflamme Instability

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Abstract

It is shown that the usual entropy argument for the Gregory-Laflamme (GL) instability for *some* appropriate black strings and p-branes gives surprising agreement up to a few percent. This may provide a strong support to the GL's horizon fragmentation, which would produce the array of higher-dimensional Schwarzschild-type's black holes finally. On the other hand, another estimator for the size of the black hole end-state relative to the compact dimension indicates a second order (i.e., smooth) phase transition for some *other* appropriate compactifications and total dimension of spacetime wherein the entropy argument is not appropriate. In this case, Horowitz-Maeda-type's non-uniform black strings or p-branes can be the final state of the GL instability.

PACS numbers: 04.70.Dy, 11.25.-w

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I. INTRODUCTION

The four-dimensional Schwarzschild black hole in Einstein gravity is well-known to be stable classically under linearized perturbations [1]. Recently, it has been shown that this extends to hold for higher dimensional cases [2]. However, Gregory and Laflamme discovered that the black strings and p-branes of 10-d low energy string theory, which have *hypercylindrical* horizons $Sch_n \times V_p$ instead of compact *hyperspherical* ones Sch_{n+p}, are found to be unstable as the compactification scale, say L, of extended directions becomes larger than the order of the horizon radius r_+–the so-called Gregory-Laflamme (GL) instability [3]. In GL's original work, they explained the instability by arguing that a black string $Sch_4 \times L$ has a lower entropy than a 5-d Schwarzschild black hole Sch_5 with the same total mass when $L > r_+$, in the context of microcanonical ensemble [1]; and they also argued that this lend support to the horizon fragmentation, which would produce array of black holes eventually. However, it is widely believed that this entropy argument for the classical stability should not be taken seriously since it *estimates* a wrong onset point of the instability–this means the black string can be classically stable even if its entropy is smaller than that of 5-d Schwarzschild black hole for some regime of L–though it provides some plausibility argument [5, 6, 7, 8]. Moreover, the GL's fragmentation scenario was disproved under very weak assumptions, including the *classical* black hole area theorem, by Horowitz and Maeda (HM) and a non-uniform black string as the final state of the GL instability [9] is considered accordingly.

In this paper, I will show that this widespread belief is *not quite* correct: If one properly apply the entropy argument to the black string solution $Sch_9 \times L$ of 10-d low energy string theory, one can estimate the onset point of the GL instability up to 2.4 % discrepancy. For p-brane solutions, the thing depends on the geometry of the compactification of p-branes. I consider two typical methods of compactifications: Thin-torus compactification and p-dimensional isotropic-torus compactification. For the former case, the discrepancy grows as p grows (n decreases) up to 35 % discrepancy for $p = 6$ ($n = 4$). But, for the latter case, the discrepancy is quite reduced up to $0.5 \sim 2.4$ %. This may provide a strong support to the GL's horizon fragmentation, which would produce the array of higher-dimensional Schwarzschild-type's black holes finally. On the other hand, another estimator for the size of the black hole end-state relative to the compact dimension indicates a second order (i.e., smooth) phase transition for some *other* appropriate compactifications and total dimension of spacetime wherein the entropy argument is not appropriate: For the black strings $Sch_n \times L$, this occurs for n as large as $n > 12$ and for the (isotropic) black p-branes $Sch_n \times L^p$, this occurs for n as low as $n < 6$ with a fixed total dimension of spacetime $d = 10$. In this case, HM-type's non-uniform black strings or p-branes can be the final state of the GL instability instead.

[1] Another explanation, based on $D - \bar{D}$ pair annihilations, is also known though it gives only the order of magnitudes [4].

II. THE BLACK STRING INSTABILITY

The black string and p-branes I am specifically interested in are those introduced by Horowitz and Strominger [10] in 10-d low energy string theory with a metric given by

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\Omega_{n-2}^2 + dx^i dx_i, \tag{1}$$

where

$$N^2 = 1 - \frac{16\pi G M_{(n)}}{(n-2)\Omega_{n-2} r^{n-3}}, \tag{2}$$

$n = 4, \ldots, 10$ and index i runs from 1 to $p = 10 - n$. $M_{(n)}$ is the mass of the n-dimensional black holes, and Ω_{n-2} is the area of the unit sphere S^{n-2} [11]. This is always a solution of the Einstein equation in $d = n + p = 10$ dimensions for compact as well as non-compact string or brane directions if the string or brane directions are completely factorized. But, this particular solution does not exist for 0-brane (i.e., 10-d black hole) and we must consider deformations of ordinary Schwarzschild solution due to non-compact dimensions in general [8, 12, 13]. But, let me approximate the 10-d black hole by the ordinary Sch_{10} metric, with 10-d radial coordinate R.

In order to compute the transition point between the black strings or branes and the 10-d black hole of the same mass due to the entropy difference, in the context of microcanonical ensemble, we need to know details of compactified dimensions. In this section let me first consider the simplest one, black string and consider simply a S^1-compactification. To this end, let me note that the masses and entropies of the black string and the 10-d black hole, with the horizon radii r_+ and R_+, are, respectively

$$M_{b.s.} = \frac{7\pi^3 r_+ L}{48G}, \quad S_{b.s.} = \frac{\pi^4 r_+^7 L}{12G},$$
$$M_{b.h.(10)} = \frac{16\pi^3 R_+^7}{105G}, \quad S_{b.h.(10)} = \frac{8\pi^4 R_+^8}{105G}.$$

Now for the same mass of the black string and the 10-d black hole, the condition of an unstable black string due to the smaller entropy than 10-d black hole is

$$L \geq \left(\frac{8}{7}\right)^8 \left(\frac{\Omega_8}{\Omega_7}\right) r_+ \approx 2.661\ r_+. \tag{3}$$

Note also that

$$L \geq \left(\frac{8}{7}\right)^7 \left(\frac{\Omega_8}{\Omega_7}\right) R_+ \approx 2.328\ R_+, \tag{4}$$

such as the 10-d black hole can easily fit in the compact dimension S^1. In terms of the wave number k for the unstable perturbation [14], (3) can be re-expressed as

$$k \leq k_S,\ \ k_S \equiv \frac{2\pi}{L_S} \approx 2.361\ r_+^{-1}, \tag{5}$$

where L_S is the entropy estimator–“equal entropy for equal mass” estimator–of the minimum length of compact dimension for the GL instability. This agrees with the GL's numerical

analysis for the classical instability under linearized perturbations $k \leq k_{GL},\ k_{GL} \approx 2.306\, r_+^{-1}$ up to 2.4 % discrepancy [2]. This good agreement is rather surprising since thermodynamic instability based on *global* entropy arguments, which have quantum origins, does not generally imply a classical instability.

III. THE BLACK p-BRANE INSTABILITY I: THIN-TORUS COMPACTIFICATION

The generalization of the string instability of the previous section to arbitrary p-branes ($2 \leq p \leq 6$) in 10-d low energy string theory requires the knowledge on the compactfication. In this section, I first consider a *thin-torus* compactification with horizons $Sch_n \times L \times V_{p-1}$ which has one compact dimension S^1 with length L and a very tiny volume $V_{p-1} \ll L^{p-1}$ for other compact dimensions. Since the effect of small compact dimensions would be tiny, I would approximate this system by the black strings $Sch_n \times L$ in $(n+1)$-dimensions effectively such as the transition problems between p-branes and 10-d black holes are reduced to those of black strings $Sch_n \times L$ and $(n+1)$-d black holes. To this end, similarly to the previous section, let me approximate the $(n+1)$-d black holes by the ordinary Sch_{n+1} metric, with $(n+1)$-d radial coordinate R. Then, the masses and entropies of the black string $Sch_n \times L$ and the $(n+1)$-d black hole are, respectively,

$$M_{b.s.(n)} = \frac{(n-2)\Omega_{n-2} r_+^{n-3} L}{16\pi G}, \quad S_{b.s.(n)} = \frac{\Omega_{(n-2)} r_+^{n-2} L}{4G},$$
$$M_{b.h.(n+1)} = \frac{(n-1)\Omega_{(n-1)} R_+^{n-2}}{16\pi G}, \quad S_{b.h.(n+1)} = \frac{\Omega_{n-1} R_+^{n-1}}{4G}.$$

Now, for the same mass of the black string and the $(n+1)$-d black hole, the condition of an unstable black string due to the smaller entropy than $(n+1)$-d black hole is

$$L \geq \left(\frac{n-1}{n-2}\right)^{n-1} \frac{\Omega_{n-1}}{\Omega_{n-2}} r_+ \equiv 2\pi\ k_S^{-1}, \tag{6}$$

where k_S is the entropy estimator of the maximum wave number for an unstable perturbation. Note also that

$$L \geq \left(\frac{n-1}{n-2}\right)^{n-2} \frac{\Omega_{n-1}}{\Omega_{n-2}} R_+ \equiv 2f\ R_+, \tag{7}$$

where $f = (\frac{n-1}{n-2})^{n-2}\Omega_{n-1}/2\Omega_{n-2}$ denotes how $(n+1)$-d black hole can fit in the compact direction S^1, such as $f \geq 1$ is required for a safe fitting. The values computed for k_S and f are listed and compared with the GL's data k_{GL} in Table I, Fig. 1 and Table II, Fig. 2, respectively. Table I and Fig. 1 show that the discrepancy grows as p grows (n decreases) up to 35 % discrepancy for $p = 6$ ($n = 4$). But, in the light of entropy argument these

[2] 2μ in GL's analysis is what I have called k [14]. And I use values of μ recently obtained by Hirayama et al. [15], which is more accurate than the original GL's analysis; I thank G. Kang for informing about this updated data. Similar data has been also obtained by E. Sorkin [16, 17] in a different context of Gubser [14] and Wiseman [18]; I thank E. Sorkin for kindly sending his data.

n	GL's data	Isotropic Torus	Thin Torus
4	0.880	0.857 (-3 %)	1.185 (+35 %)
5	1.27	1.206 (-5 %)	1.491 (+17 %)
6	1.58	1.524 (-4 %)	1.748 (+11 %)
7	1.85	1.820 (-2 %)	1.973 (+7 %)
8	2.088	2.098 (+0.5 %)	2.176 (+4.2 %)
9	2.306	2.361 (+2.4 %)	2.361 (+2.4 %)

TABLE I: Table of the entropy estimator k_S for isotropic torus and thin torus compactifications in comparison with the GL's data k_{GL}. The values in the brackets denote their discrepancies to GL's data ($r_+ \equiv 1$); the + or $-$ sign represents whether it is bigger (+) or smaller ($-$) than GL's.

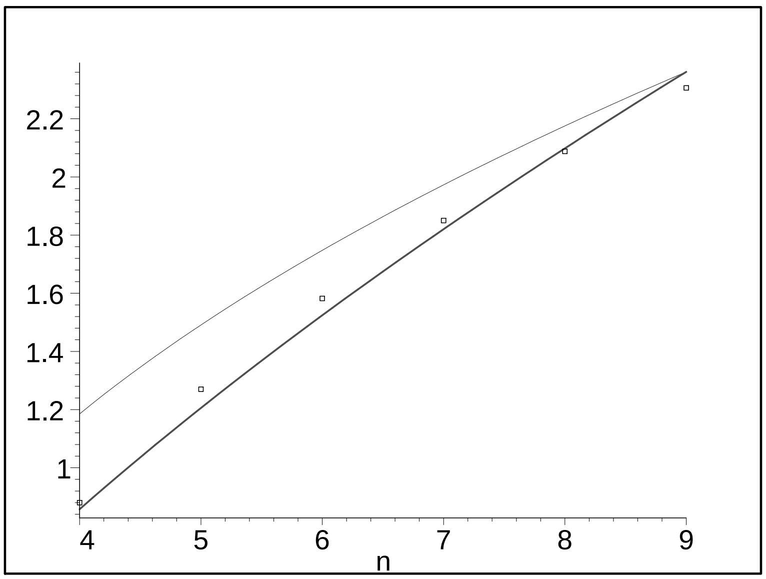

FIG. 1: Plot of k_S as a function of n for black string ($n = 9$) and p-branes ($n = 10 - p$). The boxed points represent GL's numerical data k_{GL}, the thin and the thick lines represent the values calculated for thin torus and isotropic torus compactifications, respectively.

discrepancy would only imply that the crude approximation, that I have taken, for the $Sch_{(n+1)}$ metric as the $(n+1)$-d black hole solution even with one compactified dimension S^1, and/or the thin torus limit of the compactification, which treats one specific direction differently from others, becomes bad as the dimension of the compactification p increases. So, this indicates that better approximation which treats equally all the compact directions is needed. This will be done in the next section. But, before this, let me note the followings.

First, the widespread belief [5, 6, 7, 8] that the entropy argument for the classical instability should not be taken seriously was originated from the big discrepancy of 35 % with GL's numerical analysis for the $n = 4$ case, which is found to be the worst case in the thin torus, i.e., string, approximation of the p-branes of the 10-d low energy string theory. But my analysis shows that this is not quite correct, since $n = 4$ case is not truly a black string

n	Isotropic Torus	n	Thin Torus
4	0.916	8	1.238
5	0.977 *	9	1.164
6	1.031 *	10	1.102
7	1.079	11	1.049
8	1.123	12	1.003 *
9	1.164	13	0.962 *

TABLE II: Comparison of f for isotropic torus and thin torus. The marked (*) ones are the two nearest dimensions to the critical dimension n_c.

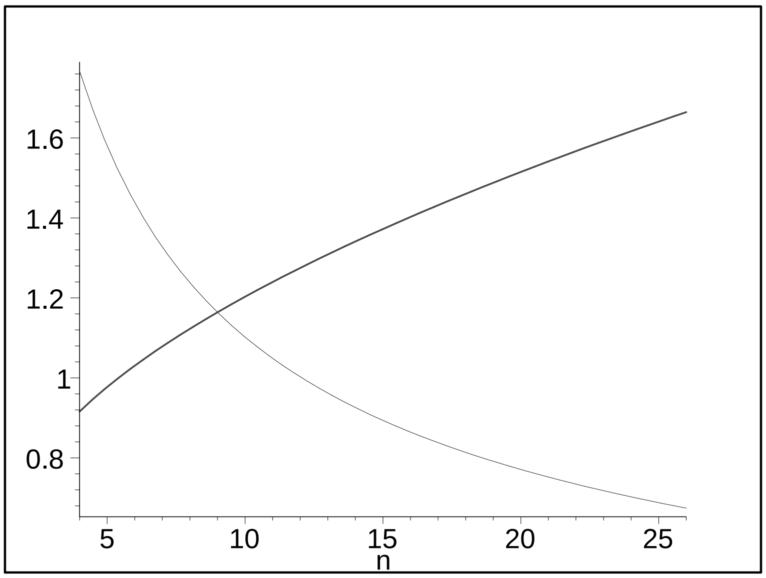

FIG. 2: Plot of f as a function of n for black string ($n = 9$) and p-branes ($n = 10-p$). The thin and the thick lines represent the values calculated for thin torus and isotropic torus compactifications, respectively. The crossing occurs at about $n = 9$.

but a 6-brane exactly [3], which would deform the black string picture quite much; one should have considered the $n = 9$ case to discuss the black string and compare with GL's data.

Second, note that the result (6) and (7) can be applied for any dimension n to analyze the transition from a black string $Sch_n \times L$ to $(n+1)$-d black hole Sch_{n+1}, though I have introduced this set-up to approximate the p-brane solutions in 10-d string theory. Then, it is interesting to observe that there is a *critical dimension* $n_c = 12$ above which $f < 1$ such as $(n+1)$-d black hole can not fit in the compact direction S^1; in this case, approximating the ordinary $Sch_{(n+1)}$ as the final state solution needs some important correction due to the compact dimension [13] such as the black string can evolve into a different final state,

[3] This is sharply contrast to the equations for the linear perturbations [3], which depend only on the sum of the Kaluza-Klein mass-squared, i.e., $\mu^2 = \sum_{i=1}^{p} \mu_i^2$ and is blind to the dimensionality p of the brane's world volumes as long as $\mu^2 \neq 0$.

presumably a non-uniform black string, between the (uniform) black string and the black hole. This indicates that the order of the phase transition between the uniform and the non-uniform black strings changes from the first (i.e., sudden transition) to second order (i.e., smooth transition) at the critical dimension n_c. Recently another estimator for the critical dimension has been considered by Sorkin [16] but one finds a very good agreement between these two estimators.

IV. THE BLACK p-BRANE INSTABILITY II: ISOTROPIC p-DIMENSIONAL TORUS COMPACTIFICATION

As a correction to the thin-torus compactification of the previous section, I will consider an isotropic p-dimensional torus compactification $Sch_n \times V_p$ where all compactified directions are treated equally. To this end, let me approximate 10-d black hole by the ordinary Sch_{10} metric, with 10-d black radial coordinate R, similarly to Sec. II. Then, the masses and entropies of a black p-brane and 10-d black hole are, respectively,

$$M_{b.b.(n)} = \frac{(n-2)\Omega_{n-2} r_+^{n-3} V_p}{16\pi G}, \quad S_{b.b.(n)} = \frac{\Omega_{(n-2)} r_+^{n-2} V_p}{4G},$$
$$M_{b.h.(10)} = \frac{16\pi^3 R_+^7}{105G}, \quad S_{b.h.(10)} = \frac{8\pi^4 R_+^8}{105G}.$$

Now, for the same mass of the black p-branes and the 10-d black hole, the condition of an unstable black p-branes due to the smaller entropy than 10-d black hole is

$$V_p \geq \left(\frac{8}{n-2}\right)^8 \frac{\Omega_8}{\Omega_{n-2}} r_+^{10-n} \tag{8}$$

while

$$V_p \geq \left(\frac{8}{n-2}\right)^{n-2} \frac{\Omega_8}{\Omega_{n-2}} R_+^{10-n}. \tag{9}$$

Moreover, since I am considering a p-dimensional torus with equal length $L = (V_p)^{1/p}$, (8) and (9) can be re-expressed, in terms of L and the associated entropy estimator of the maximum wave number k_S for the unstable perturbation, as

$$L \geq \left(\frac{8}{n-2}\right)^{\frac{8}{10-n}} \left(\frac{\Omega_8}{\Omega_{n-2}}\right)^{\frac{1}{10-n}} r_+ \equiv 2\pi\ k_S^{-1} \tag{10}$$

and

$$L \geq \left(\frac{8}{n-2}\right)^{\frac{8}{10-n}} \left(\frac{\Omega_8}{\Omega_{n-2}}\right)^{\frac{1}{10-n}} R_+ \equiv 2f\ R_+. \tag{11}$$

The values computed for k_S and f are listed and plotted also in Table I, Fig. 1 and Table II, Fig. 2, respectively, in comparison with GL's data k_{GL} and the results for the thin-torus compactfication. Table I and Fig. 1 show that the discrepancy of the thin torus approximation have been quite reduced in the isotropic torus and the worst one is about $4 \sim 5$ % for $p = 4, 5$ $(n = 6, 5)$; all the other cases have been less than about $2 \sim 3$ % and

the best one is 0.5 % for $p = 2$ ($n = 8$); moreover, compared to the growing discrepancy for the thin-torus as p grows (n decreases), that of the isotropic torus is almost stable such as this improved approximation is fairly good.

On the other hand, according to the result for f in Table II and Fig. 2, there is a critical dimension $n_c = 6$ below which $f < 1$ such as the 10-d black hole can not fit in the compact dimension S^1. This implies that approximating the ordinary Sch_{10} as the final state of the black p-branes needs some important corrections due to the compactified dimension such as the uniform black p-brane can evolve into a different final state, presumably a non-uniform black p-brane. So, the relatively big discrepancies for $n = 5, 6$ would not be so surprising in the light of entropy argument; but it is a remarkable fact that $n = 4$ case has a relatively good agreement with GL's data with 3 % discrepancy even though it does not have to be. Hence, by taking into account this additional fact to the result of k_S in Table I and Fig. 1, the true discrepancy in this approximation would be quite smaller and the reliable results would have discrepancy only about 0.5 % $\sim$ 2.4 % by excluding $n = 4, 5, 6$ cases.

Furthermore, this also indicates a smooth decay of an unstable (uniform) black p-brane $Sch_n \times L^p$ to the non-uniform state for n as low as 4 or 5. This is in contrast to the decay of black string, where n as large as $n > 12$ is required for a smooth decay. More recently another estimator for the critical dimension, following Sorkin [16], has been also considered by Kol and Sorkin [19] and they found a very good agreement with mine again; moreover they found interestingly that $d = 10$ is the smallest total dimension of the spacetime to allow a smooth decay of an unstable black brane to the non-uniform state.

V. DISCUSSION

I have shown that the usual entropy argument for the GL instability for *some* appropriate black strings and p-branes gives surprising agreement up to a few percent. This may provide a strong support to the GL's horizon fragmentation, which would produce the array of–single in my analysis–higher dimensional Schwarzschild-type's black holes finally; this result is remarkable in that the end point of the unstable evolution, which is by its nature "non-linear", crucially affects the onset of the instability calculation, which is by its nature "linear".

On the other hand, another estimator for the size of the black hole end-state relative to the compact dimension indicates a second order (i.e., smooth) phase transition for some *other* appropriate compactifications and total dimension of spacetime wherein the entropy argument is not appropriate. For the black strings $Sch_n \times L$, this occurs for n as large as $n > 12$ and for the (isotropic) black p-branes $Sch_n \times L^p$, this occurs for n as low as $n < 6$ with a fixed total dimension of spacetime $d = 10$. In this case, HM-type's non-uniform black strings or p-branes can be a natural final state of the GL instability. This result agrees quite well with the analysis of Kol and Sorkin wherein different estimator has be considered [16, 19].

Note added: After the first appearance of this paper, I was informed by E. Sorkin that my analysis of the instability for thin-torus compactification is very similar to that of Ref. [16] which uses a single dimensionless parameter $\tilde{\mu} \equiv GM/L^{n-2}$ instead of r_+, R_+, L. Afterward, I have checked that his result (9) on the critical values of $\tilde{\mu}$ for the onset of an instability agrees exactly with my result (6). And his analysis [16, 17] on the black string perturbation shows a quite good agreement with the thin-torus set-up up to $0.2 \sim 1\%$ discrepancy for $10 \leq n \leq 12$, in contrast to $n \leq 9$, where the isotropic-torus set-up is more favorable (

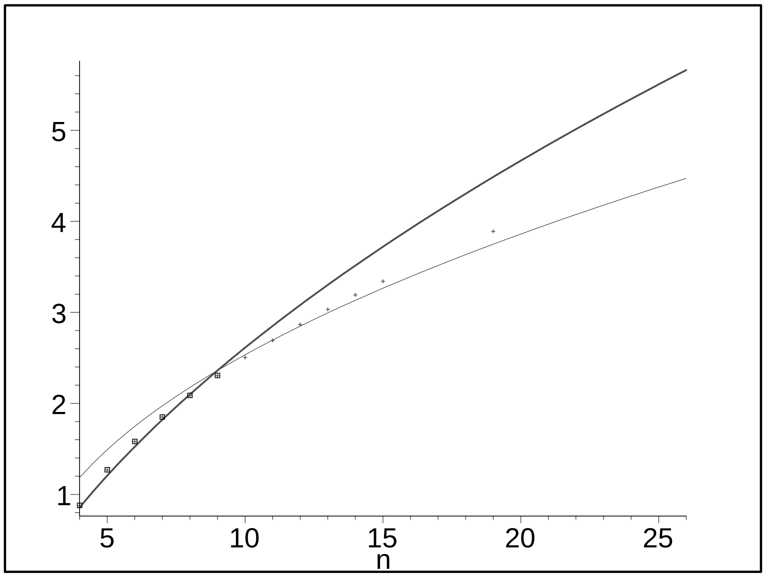

FIG. 3: Plot of k_S as a function of n for Sorkin's black string analysis in arbitrary dimensions, in comparison with Fig.1; cross points correspond to Sorkin's data [16, 17] and these overlap with GL's data for $n \leq 9$.

Fig. 3); this indicates *another critical dimension* at $n = 9$ which agrees with Kol's "merger point", where the string and black hole branches merge [20]. The increasing discrepancy above $n = 12$ is not so surprising since this is the regime where the naive thin-torus set-up does not have to be correct, due to $f < 1$, such as the black string can evolve into a different final state, presumably a non-uniform black string, as in $n < 6$ cases of isotropic-torus set-up in Sec. IV.

Acknowledgments

I would like to thank Ulf Danielsson, James Gregory, Steven Gubser, Troels Harmark, Gary Horowitz, Akihiro Ishibashi, Gungwon Kang, Hideo Kodama, Luis Lehner, and Evgeny Sorkin for useful correspondences. I was supported by the Korea Research Foundation Grant (KRF-2002-070-C00022).

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